Ordered Ramsey Numbers of Small Graphs

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MIT PRIMES
Under William Kuszmaul and Jacob Fox

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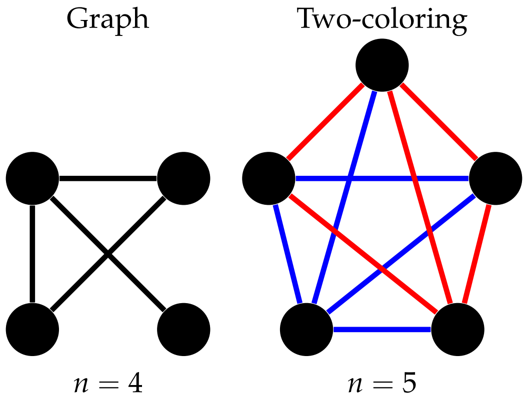
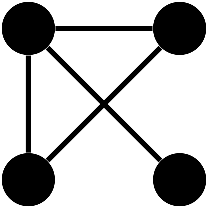
Graphs and 2-colorings on n vertices



2-COLORINGS CAN CONTAIN GRAPHS

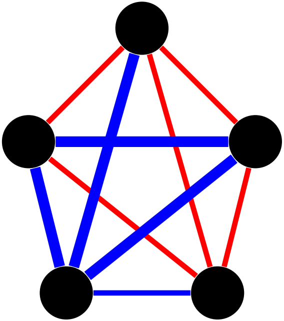
Graph



$n = 4$

is contained in

Two-coloring



$n = 5$

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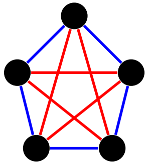
- Not all 2-colorings on 5 vertices do:



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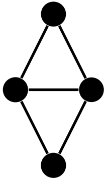
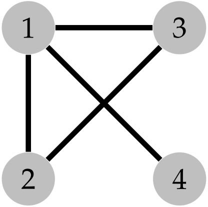
2. Results for specific small graphs
   Example: $R(\text{diamond graph}) = 10$



Ordered graphs and 2-colorings on n vertices

Ordered graph



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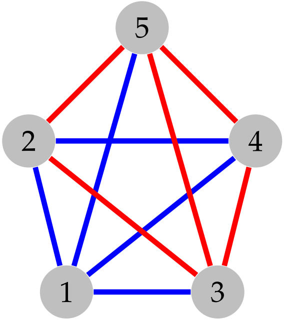
Ordered two-coloring



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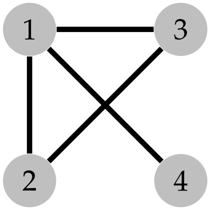
Ordered 2-colorings can contain ordered graphs

Ordered graph



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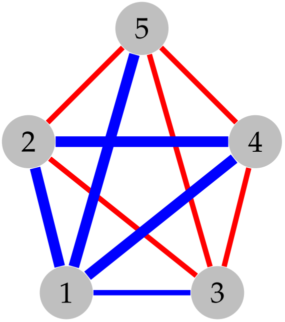

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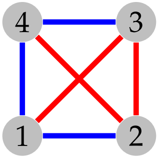

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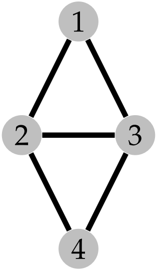
2. **Our Research Goal:** Results for specific small graphs

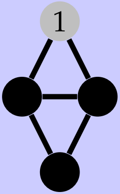
Our research

We want to find the ordered Ramsey number of the standard ordering of the diamond graph (*DG*).



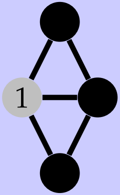
Work towards upper bound

Theorem

$R_<\Big($ 

$\Big) \leq 14$

Theorem

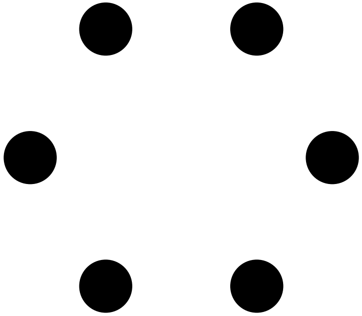
$R_<\Big($ 

$\Big) \leq 13$

Single-vertex anchoring

Upper bound proofs for unordered Ramsey numbers often center around a particular vertex.

Example: $R(\Delta) \leq 6$



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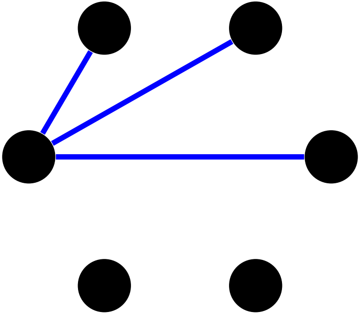

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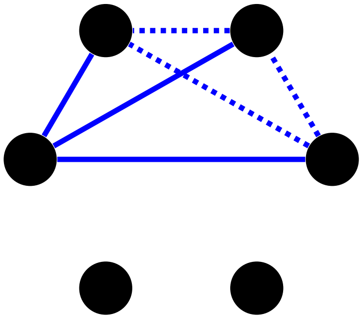

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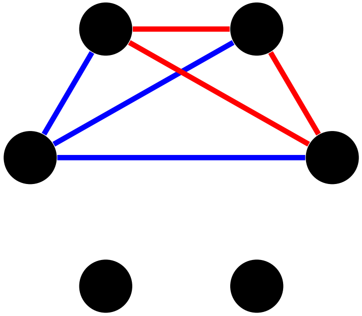
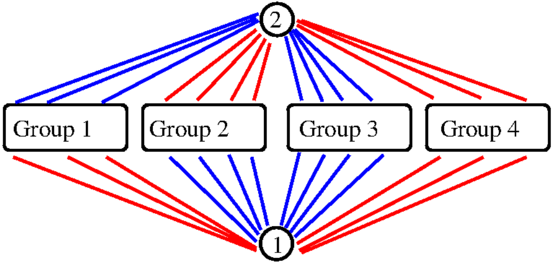


An idea for ordered Ramsey numbers: Two-vertex anchoring

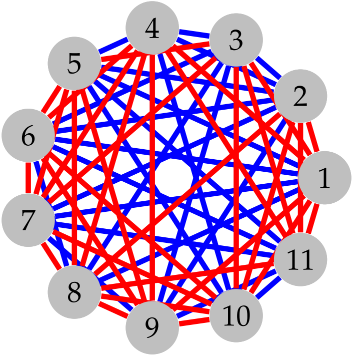
To get bounds for ordered Ramsey numbers, we anchor our proofs at two vertices.



A Lower bound

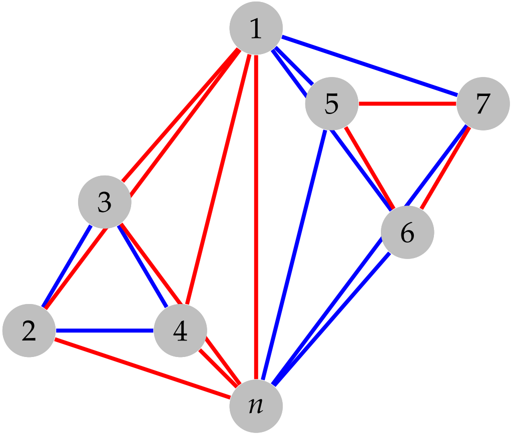
Theorem

$$R_{<}(DG) \geq 12$$



Using a computer to get a lower bound

First, build a skeleton using two-vertex anchoring



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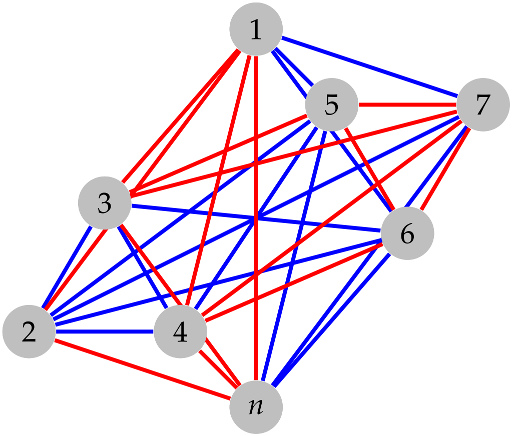
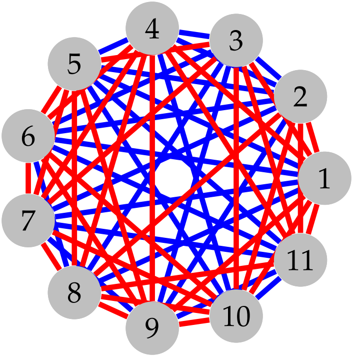


Using a computer to get a lower bound

Next, fill in the rest of the two-coloring by force.

Theorem

$$R_{<}(DG) \geq 12$$



Future work

- Tighten bounds and extend upper bounds to full ordering of DG.
- Find ordered Ramsey numbers of other small graphs.
- Find asymptotic growth rate of ordered Ramsey numbers of P_n^k, an important family of ordered graphs whose smallest interesting member is DG.

Acknowledgments

- William Kuszmaul for providing so much valuable guidance and being an overall great mentor.
- Prof. Jacob Fox for suggesting the project and providing directions of research.
- MIT PRIMES for the opportunity to conduct this research.

References

1. M. Balko, J. Cibulka, K. Král, and J. Kynčl. Ramsey numbers of ordered graphs. *Electronic Notes in Discrete Mathematics*, 49:419–424, 2015.
2. J. Bondy and P. Erdös. Ramsey numbers for cycles in graphs. *Journal of Combinatorial Theory, Series B*, 14(1):46–54, 1973.
3. V. Chvátal and F. Harary. Generalized Ramsey theory for graphs. ii. small diagonal numbers. *Proceedings of the American Mathematical Society*, 32(2):389–394, 1972.
4. D. Conlon, J. Fox, C. Lee, and B. Sudakov. Ordered Ramsey numbers. *arXiv preprint arXiv:1410.5292*, 2014.